

What is the meaning of non-uniqueness of FRW and Schwarzschild metrics?

Leonid V. Verozub,
Kharkov National University
lverozub@gmail.com

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Abstract

It is shown that any theory of gravitation, based on the hypothesis of the geodesic motion of test particles must be invariant under geodesic (projective) mappings of the used space-time. The reason is that due to invariance of the equations of geodesic lines under a continuous group of transformations of the coefficients of affine connection, there is a wide class of transformations of the geometrical objects of Riemannian space-time which leaves invariant the equations of motion of test particles. The FRW metric in cosmology and the Schwarzschild metric are a good example to make sure that the standard space-time metrics does not determine the gravitational field unequivocally.

1 Introduction

The equations of motion of test bodies play a fundamental role in the classical field theory. They give evidence for the existence of the field and allow us to find its properties. The functions appearing in these equations are characteristics of the field.

In the case of gravity, the gravitational equations of motion of test particles are invariant under some group of transformations of the Christoffel symbols $\Gamma_{\beta\gamma}^{\alpha}$ – of geodesic transformations [1]:

$$\bar{\Gamma}(x)_{\beta\gamma}^{\alpha} = \Gamma(x)_{\beta\gamma}^{\alpha} + \psi_{\beta}(x)\delta_{\gamma}^{\alpha} + \psi_{\gamma}(x)\delta_{\beta}^{\alpha}, \quad (1)$$

where $\psi_{\alpha}(x)$ is an arbitrary gradient covector field. It is easiest to see it, if the coordinate $t = x^0/c$ is used as a parameter in geodesic line:

$$\ddot{x}^{\alpha} + (\Gamma_{\beta\gamma}^{\alpha} - c^{-1}\Gamma_{\beta\gamma}^0\dot{x}^{\alpha})\dot{x}^{\beta}\dot{x}^{\gamma} = 0. \quad (2)$$

It seems obvious that the Christoffel symbols $\bar{\Gamma}_{\beta\gamma}^{\alpha}(x)$ and $\Gamma_{\beta\gamma}^{\alpha}(x)$ describe the same physical gravitational field, just as the 4-potentials in classical electrodynamics, connected by a gauge transformation $A_{\beta} \rightarrow A_{\beta} + \partial_{\beta}\phi(x)$.

Such a gauge transformation of the Christoffel symbols induces a corresponding transformation of the metric tensor $g_{\mu\nu}$, the curvature tensor $R_{\alpha\beta\gamma}^{\delta}$, and the Ricci tensor $R_{\alpha\beta}$. So, all these objects in themselves have no more physical

meaning than 4-potentials A_μ in electrodynamics. Evidently due to this fact the equations of gravitational field, based on the hypothesis on free motion on geodesic lines, must be invariant under the geodesic (projective) transformations. It is well known that even vacuum Einstein's equations do not satisfy this condition [2]¹.

In Riemannian space-time, eqs.(1) are equivalent to the mappings of the space-time V with the metric tensor $g(x)_{\alpha\beta}$ to the space-time \bar{V} with the metric tensor $\bar{g}(x)_{\alpha\beta}$ defined by the following PDE:

$$\bar{g}(x)_{\alpha\beta;\gamma} = 2\psi(x)_\gamma \bar{g}_{\alpha\beta}(x) + \psi(x)_\alpha \bar{g}_{\gamma\beta}(x) + \psi(x)_\beta \bar{g}_{\alpha\gamma}(x), \quad (3)$$

where a semicolon denotes a covariant derivative with respect to x^α in V . There exists extensive literature on the investigation of the possibility of geodesic mapping $V \rightarrow \bar{V}$ based on these equations [4].

Consequently, every solution of Einstein's equations in any coordinate system gives in general only one of many physically equivalent metrics [5] .

2 FRW metric

It is recently this fact has been discovered independently in [6] for the case of the FRW metric. It was noted in this paper (and after that in [7]) that the line element of this cosmological model admits one-parameter transformations of the metric tensor that leaves unchanged non parameterized geodesics.

But the truth is that this is not a random fact. This is not a specifics of the FRW metric. The truth is that the metric, Christoffel symbols, or the curvature tensor define the gravity field only up to geodesic transformations, which should play the role of gauge transformations in any geometrical theory of gravitation[5].

Due to simplicity of the FRW metric, consideration of the consequences of such geodesic equivalence of metrics is especially simple.

Consider the line elements of a Riemannian space-time V :

$$ds^2 = b(t) dt^2 + a(t) \sigma_{ik}(x^1, x^2, x^3) dx^i dx^k. \quad (4)$$

It is known [3] that geodesics of such metric are the same as the ones of the space-time \bar{V} with the line element

$$\bar{ds}^2 = B(t) dt^2 + A(t) \sigma_{ik}(x^1, x^2, x^3) dx^i dx^k. \quad (5)$$

where

$$B(t) = \frac{b(t)}{[1 + q a(t)]^2}, \quad (6)$$

$$A(t) = \frac{a(t)}{1 + q a(t)}, \quad (7)$$

and q is an arbitrary constant .

Consider briefly the proof of this important fact.

¹We mean the original Einstein's equations, and not frequently considered its mathematical generalization

Contracting (2) with respect to α and β , we obtain $\bar{\Gamma}_{\beta\gamma}^{\beta} = \Gamma_{\beta\gamma}^{\beta} + (n+1)\psi_{\beta}$. Consequently,

$$\psi_{\beta} = \frac{1}{2(n+1)} \frac{\partial}{\partial x^{\beta}} \ln \left| \frac{\det \bar{g}}{\det g} \right|, \quad (8)$$

which shows that in the case under consideration only 0-component of ψ_{α} is other than zero.

The useful for us components of the Christoffel symbols of V are:

$$\bar{\Gamma}_{00}^0 = b'(t)/2b, \quad \bar{\Gamma}_{11}^0 = a'(t)/2b(t), \quad \bar{\Gamma}_{10}^1 = a'(t)/2a(t),$$

and the same components of \bar{V} :

$$\bar{\Gamma}_{00}^0 = B'(t)/2B, \quad \bar{\Gamma}_{11}^0 = -A'(t)/2B(t), \quad \bar{\Gamma}_{10}^1 = A'(t)/2A(t). \quad (9)$$

Then eqs. (1) gives the following equations

$$\frac{A'(t)}{A(t)} - \frac{a'(t)}{a(t)} = 2\psi_0, \quad (10)$$

$$\frac{B'(t)}{B(t)} - \frac{b'(t)}{b(t)} = 4\psi_0, \quad (11)$$

$$\frac{A'(t)}{B(t)} - \frac{a'(t)}{b(t)} = 0. \quad (12)$$

Therefore, $A/a = \exp(2 \int \psi(t)_0 dt)$, $B/b = \exp(4 \int \psi(t)_0 dt)$ where the integration constants are equal to 1 because at $\psi(t)_0 = 0$ the functions $A(t) = a(t)$ and $B(t) = b(t)$. Consequently, $B(t)/b(t) = (A(t)/a(t))^2$, and with (12) we obtain the differential equations

$$A'(x) - A(t)^2 \frac{a'(t)}{a(t)} = 0, \quad (13)$$

which gives (7). Now from previous equation we obtain the function $B(t)$ in the form (6).

On the contrary, if in eq. (1) to set $\psi_i = 0$ for $i=1,2,3$, and $\psi_0 = -1/2 \partial \ln(1 + qb(t))/\partial t$, then eqs. (10), (11), and (12) are satisfied. Thus, with this choice of the covector field $\psi(x)_{\alpha}$, the line element (4) at $b = -1$ is equivalent to (5). In other words, the both line elements have the same equations of motion of test particles.

3 Schwarzschild metric

As another example, we show here that a static centrally symmetric metric

$$ds^2 = b(r)dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) - a(r)dt^2, \quad (14)$$

(in particular, Shvartsschild metric) is not unique. Namely, in a given coordinate system it has common geodesic lines with a metric of the form

$$\overline{ds}^2 = B(r) dr^2 + F(r)^2 (d\theta^2 + \sin^2 \theta d\phi^2) - A(r) dt^2, \quad (15)$$

where $A(x), B(x)$ and $F(x)$ are functions of r , depending on a continuous parameter.

The Christoffel symbols for (14) is given by

$$\Gamma_{rr}^r = \frac{1}{2} \frac{b'(r)}{b(r)}, \quad \Gamma_{r\theta}^\theta = \frac{1}{r}, \quad \Gamma_{r\phi}^\phi = \frac{1}{r}, \quad \Gamma_{rt}^t = \frac{1}{2} \frac{a'(r)}{a(r)}, \quad \Gamma_{\theta\theta}^r = -\frac{r}{b(r)}, \quad (16)$$

$$\Gamma_{\theta\phi}^\phi = \frac{\cos \theta}{\sin \theta}, \quad \Gamma_{\phi\phi}^r = -\frac{r \sin^2 \theta}{b(r)}, \quad \Gamma_{\phi\phi}^\theta = -\sin \theta \cos \theta, \quad \Gamma_{tt}^r = \frac{1}{2} \frac{a'(r)}{b(r)}. \quad (17)$$

The Christoffel symbols for 15 are:

$$\overline{\Gamma}_{rr}^r = \frac{1}{2} \frac{B'(r)}{B(r)}, \quad \overline{\Gamma}_{r\theta}^\theta = \frac{F'(r)}{F(r)}, \quad \overline{\Gamma}_{r\phi}^\phi = \frac{F'(r)}{F(r)}, \quad \overline{\Gamma}_{rt}^t = \frac{1}{2} \frac{A'(r)}{A(r)}, \quad \overline{\Gamma}_{\theta\theta}^r = -\frac{F(r) F'(r)}{B(r)}, \quad (18)$$

$$\overline{\Gamma}_{\theta\phi}^\phi = \frac{\cos \theta}{\sin \theta}, \quad \overline{\Gamma}_{\phi\phi}^r = -\frac{F(r) F'(r) \sin^2 \theta}{B(r)}, \quad \overline{\Gamma}_{\phi\phi}^\theta = -\sin \theta \cos \theta, \quad \overline{\Gamma}_{tt}^r = \frac{1}{2} \frac{A'(r)}{B(r)}, \quad (19)$$

where a prime here and later denotes a derivative with respect to r .

In view of this, Levi-chevita equations (1) yields:

$$\frac{B'(r)}{B(r)} - \frac{b'(r)}{b(r)} = 4\psi_r(r), \quad F(r)F'(r)b(r) - rB(r) = 0, \quad A'(r)b(r) - a'(r)B(r) = 0, \quad (20)$$

$$\frac{F'(r)}{F(r)} - \frac{1}{r} = \psi_r(r), \quad \frac{A'(r)}{A(r)} - \frac{a'(r)}{a(r)} = 2\psi_r(r), \quad \psi_\theta(r) = \psi_\phi(r) = \psi_t(r) = 0. \quad (21)$$

According to (8) the function $\psi_r(r)$. can be written as

$$\psi_r = \partial \ln \chi / \partial r,$$

where

$$\chi(r) = \left(\frac{\bar{g}}{g} \right)^{1/2(n+1)}.$$

Consequently,

$$\begin{aligned} B &= b\chi^4; & A &= a\chi^2; & F &= \chi; \\ A' &= a'\chi^4; & (F^2)' &= 2r\chi^4; \end{aligned}$$

Formulas for the function $F(r)$ are compatible only if the functions $\chi(r)$ are the solution of the differential equations

$$r\chi'(r) + \chi(r) - \chi(r)^3 = 0$$

which yields

$$\chi(r) = (1 + kr^2)^{1/2},$$

where k is an arbitrary constant.

As a result, formulas which express the $A(r)$, $B(r)$, and $F(r)$ by $a(r)$ and $b(r)$ are given by

$$A(r) = \frac{a(r)}{1 + kr^2}, \quad B(r) = \frac{b(r)}{(1 + kr^2)^2}, \quad F(r) = \frac{r}{(1 + kr^2)^{1/2}}, \quad (22)$$

where k is a constant satisfying appropriate physical conditions.

4 Discussion

It is obvious that the geodesics (projective) mappings of Riemannian spaces should be considered as gauge transformations of the differential equation, which is used to determine the geometrical characteristics of gravity in any theory based on Einstein's hypothesis of the motion of test bodies along geodesics of Riemannian space.

The fact that the connection coefficients and the metric tensor are determined up to an arbitrary geodesic mapping, does not mean that our physical space-time has a projective symmetry. No doubt the physical space-time is locally pseudo-Euclidean and the notion of length has a physical sense. This fact means that the geometric characteristics of the physical space-time (the coefficient of the connection or metric tensor) can not be directly identified with the characteristics of the gravitational field, they are not observable variables of the field. Such variables must be geodesically invariant.

However, the possibility to define observable variables of gravitational field exists. For example, although Christoffel symbols are not viewed as the observable characteristics of gravity, there are symbols of Thomas, which are geodesically invariant objects. They are not tensors. However, in the presence of a flat background metric, a tensor object from the symbols of Thomas can be formed [5].

As for metrics, there are two possibilities to compare such theory with observations.

Firstly, we can use solution of the field equations at some selected gauge condition, just as we do it with solutions of the Einstein equations at a selected coordinate conditions.² (It is used in [5]).

Secondly, there is an object that is geodesically invariant generalization of the metric tensor $g_{\alpha\beta}$ if we consider the metric tensor as the 4-components of some 5-dimensional tensor in the spirit of a 5-dimensional interpretation of geodesic maps dating back to Thomas [9] and Veblen [12].

Let X^A ($A = 0 \div 4$) be homogeneous coordinates of points in the tangent space of the space-time manifold with an arbitrary factor, which is conveniently labeled as $\exp(x^4)$. Then, in addition to coordinate transformations

$$\overline{x}^\alpha = \overline{x}^\alpha(x^0, x^1, x^2, x^3) \quad (23)$$

²From a fundamental point of view, the problem of observables in general relativity has not been solved.

we must also take into account the change of this factor by transformation of the fifth coordinate

$$\bar{x}^4 = x^4 + \log \rho \quad (24)$$

where ρ is an arbitrary function of x^α . In this auxiliary five-dimensional manifold, we can define the geometric objects, which are transformed through (23) and (24). In particular, a tensor transforms as follows

$$\bar{Q}_{AB} = Q_{CD} \frac{\partial x^C}{\partial x^A} \frac{\partial x^D}{\partial x^B},$$

where capital letters range from 0 to 4.

The equation

$$G_{AB} X^A X^B = 0$$

defines the quadric for which the equation of light cone $g_{\alpha\beta} dx^\alpha dx^\beta = 0$ is an asymptotic. Tensor G_{AB} determines a metric. In this case, $g_{AB} = G_{AB}/G_{44}$ is a projective tensor, such that $g_{44} = 1$. If we define $f_A = g_{A0}$, it follows from the transformation law of G_{AB} , that f_A is a covariant projective vector which transforms under pure 4-transformation of coordinates as

$$\bar{f}_4 = f_4; \bar{f}_\alpha = f_\beta \frac{\partial x^\beta}{\partial \bar{x}^\alpha},$$

and under pure transformation of 5-coordinate (i.e under projective transformations) as

$$\bar{f}_4 = f_4, \bar{f}_\alpha = f_\alpha - \frac{\partial \log \rho}{\partial x^\alpha}.$$

The tensor g_{AB} is of the form

$$g_{AB} = \begin{pmatrix} 1 & f_\alpha \\ f_\beta & g_{\alpha\beta} \end{pmatrix} \quad (25)$$

where $g_{\alpha\beta}$ is an affine tensor which can be identified with the metric tensor of space-time. It follows from the transformation law (23) and (24) that the object

$$\bar{g}_{\alpha\beta} = g_{\alpha\beta} - f_\alpha f_\beta \quad (26)$$

is invariant under the transformation of the fifth coordinate, and hence is invariant under the geodesic (projective) maps of space-time.

What is a field of $f_\alpha(x)$? It follows from the transformation properties of the vector f_α with respect to the transformation of 5-th coordinate that it can be a kind of a gradient invariant physical field similar to the 4-potential of electromagnetic field which was used in the Kaluza-Klein model. However, this vector can also be formed from the components of the metric tensor, since the Christoffel symbols $\Gamma_\beta = \Gamma_{\alpha\beta}^\alpha$ have the same transformation law under the transformation of 5-th coordinate. This possibility was used in paper [11].

From this point of view Einstein's equation is very similar to some gauge-invariant equations that describe gravity in a fixed gauge. The simplest equation of this kind, which do not contradict the available observations, are proposed in [5].

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